HEAT TRANSFER IN A LAMINAR JET AT A WALL

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The consideration of the heating of media within the framework of the thermal conductivity theory becomes incorrect if the change in the temperature takes place over broad limits; this leads to the need to take account of the dependence of the coefficient of thermal conductivity on the temperature. For example, such a dependence holds when radiative heat transfer plays a significant role in the mechanism [1].

If the thermal-conductivity coefficient depends on the temperature according to an exponential law, then the propagation of thermal perturbations in a medium with a zero temperature takes place with a finite velocity of the thermal wave front motion [1]. If, in addition to this, there are heat sources in the medium, then the thermal wave front may remain fixed with respect to the source of the thermal perturbations [2]. It is shown below that an analogous phenomenon of the spatial localization of thermal perturbations can be observed also with a solution of problems of a steady-state temperature boundary layer; this fact is a result of the combined effect of the exponential dependence of the thermal-conductivity coefficient on the temperature in a moving medium.

Let us consider the problem of the steady-state distribution of the temperature in a submerged laminar jet, propagating along a solid plane heated surface, whose temperature T_W = const > 0. Let a flat jet of liquid, having a temperature T = 0, issue in the direction of the x axis from a narrow slit x = 0, y = 0 into a half-space y > 0, filled with the same liquid (Fig. 1). It is postulated that the thermal conductivity of the liquid depends on the temperature according to an exponential law, and that the remaining parameters of the liquid are constant.

Neglecting viscous dissipation in the liquid, the system of heat- and mass-transfer equations, describing the above process in a boundary-layer approximation, is written in the form [3]

$$\begin{gathered}\partial u/\partial x+\partial v/\partial y=0,\\ u\partial u/\partial x+v\partial u/\partial y=\nu\partial^2u/\partial y^2,\\ u\partial T/\partial x+v\partial T/\partial y=a\partial^2T^n/\partial y^2,\end{gathered}\tag{1}$$

where a = const > 0; n = const > 0; ν is the viscosity of the liquid; anT^{n-1} is the thermal-conductivity coefficient of the liquid.

The projections of the velocity of the liquid u(x, y) and v(x, y) are determined independently of the values of the temperature T(x, y); the corresponding problem is self-similar and has the solution [3, 4]

$$\begin{gathered}u(x,y)=F'(\eta)\left(\frac{E}{\nu x}\right)^{1/2},\\ v(x,y)=\frac{1}{4}(3\eta F'(\eta)-F(\eta))\left(\frac{\nu E}{x^3}\right)^{1/4},\end{gathered}\tag{2}$$

where $\eta = y(E/\nu^3x^3)^{1/4}$ is a self-similar variable; $E=\int\limits_0^\infty u\left(\int\limits_y^\infty u^2dy\right)dy=\text{const}$ is the retained invariant of the problem; the function F(η), a curve of which is shown in Fig. 2, is determined implicitly (see, for example, [3]). In Fig. 1, the flow lines are solid, and the lines of η = const are dashed.

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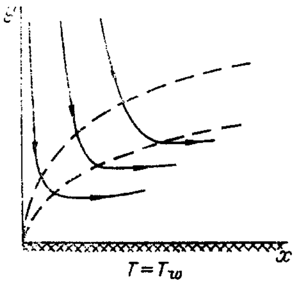

Fig. 1

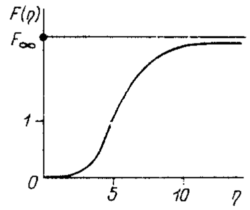

Fig. 2

The distribution of the temperature T(x, y) in the prewall jet must be determined with a solution of the third equation of system (1), i.e., the thermal-conductivity equation with the obvious boundary conditions

$$T(x, 0) = T_w, \; T(x, \infty) = 0. \tag{3}$$

The sought function T(x, y) and the derivatives $\partial T^n/\partial x$, $\partial T^n/\partial y$ must be continuous everywhere with x, y ≥ 0, which corresponds to the continuity of the temperature and the heat flux $q = -a\nabla T^n$. In addition, the physically obvious condition of the reversion of the heat flux to zero with $y \to \infty$ must also be satisfied

$$(\partial T^n/\partial x)(x, \infty) = (\partial T^n/\partial y)(x, \infty) = 0. \tag{4}$$

Using the usual premises of the theory of dimensionality, we shall seek the self-similar solution T(x, y) of the problem (1)-(4) in the form $T(x, y) = T_w\theta(\eta)$. Here the function $\theta(\eta)$ is a solution of the following problem:

$$\begin{gathered} -F(\eta)\theta'(\eta) = (4/n\sigma_w)[\theta^n(\eta)]'', \\ \theta(0) = 1, \; \theta(\infty) = (\theta^n)'(\infty) = 0, \end{gathered} \tag{5}$$

where $\sigma_w = \nu/naT_w^{n-1}$ is the effective Prandtl number in the liquid near the heated wall.

In the linear case with n = 1, problem (5) was discussed in [5]; with n ≠ 1, Eq. (5) can be integrated only numerically; however, it is advisable to make a preliminary investigation of the analytical structure of its solution. In view of this, we consider the auxiliary problem

$$-F_0\tilde{\theta}'(\eta) = (4/n\sigma_w)[\tilde{\theta}^n(\eta)]'', \; \tilde{\theta}(0) = 1, \tilde{\theta}(\infty) = (\tilde{\theta}^n)'(\infty) = 0, \tag{6}$$

differing from problem (5) in that the function F(η) is replaced by a constant F_0.

We note that the function F(η) varies only slightly with $\eta > \eta_c \approx 10$ $[F(\eta) \geq \eta_c) \approx F_\infty]$; therefore, for values of x satisfying the condition

$$\eta_c(Ex/\nu^3)^{3/4} \ll 1, \tag{7}$$

the solution $\tilde{T}(x, y) = T_w\tilde{\theta}(\eta)$, obtained as the result of a solution of the auxiliary problem (6) with $F_0 = F_\infty$, differs very little from the exact solution T(x,y) of problem (5). Figure 3 gives profiles of the temperature along y: a) for values of x_1 and x_2 satisfying the condition $Ex_1/\nu^3 = 0.01$; $Ex_2/\nu^3 = 0.02$; n = 1.5; $\sigma_w = 1$ [the exact solution of problem (5) is shown by a dashed line]; b) for different value of n with $Ex/\nu^3 = 0.01$; $\sigma_w = 1$.

One integration of Eq. (6), taking account of the conditions with $\eta \to \infty$, leads to the equation

$$-(1/4)(n-1)\sigma_w F_0 = [\tilde{\theta}^{n-1}(\eta)]', \tag{8}$$

whose partial solution, satisfying the condition $\tilde{\theta}(0) = 1$, has the form

$$\tilde{\theta}(\eta) = \left[1 - \frac{1}{4}(n-1)\sigma_w F_0 \eta\right]^{\frac{1}{n-1}}. \tag{9}$$

With n < 1, the function $\tilde{\theta}(\eta)$ (9) is determined everywhere with $0 < \eta < \infty$, and $\tilde{T}(x, y)$ is written in the form

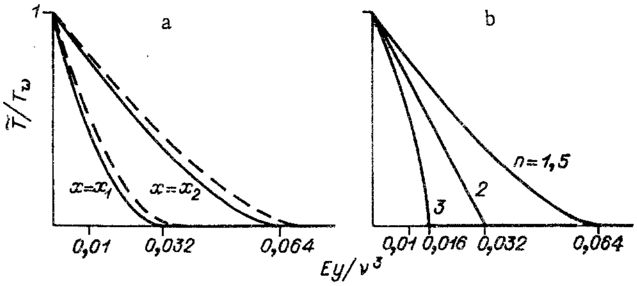

Fig. 3

$$\tilde{T}(x, y) = T_w\left[1 + \frac{1}{4}(1-n)\sigma_w F_0\left(\frac{E}{\nu^3 x^3}\right)^{1/4} y\right]^{\frac{1}{n-1}},$$
$$x, y \geqslant 0, \quad n < 1. \tag{10}$$

If n > 1, the function $\tilde{\theta}(\eta)$ reverts to zero with

$$\eta = \tilde{\eta}_f = 4/(n-1)\sigma_w F_0.$$

Equation (8) with n > 1 has the singular solution $\tilde{\theta} = 0$; therefore, the solution of problem (6) with n > 1 must be regarded as a generalized solution, made up at the point $\eta = \tilde{\eta}_f$ of the partial solution (9) with $0 \leq \eta \leq \tilde{\eta}_f$ and the singular solution $\tilde{\theta} = 0$ with $\eta_f \leq \eta < \infty$ (see Fig. 3).

The generalized solution, constructed in the above manner, generally speaking, has a weak discontinuity at the point $\eta = \tilde{\eta}_f$ [the derivatives $d^{m+1}\theta/d\eta^{m+1}$ undergo a discontinuity at the point $\eta = \tilde{\eta}_f$, where m is the greatest whole number such that $m < 1/(n-1)$]. For example, with n = 1.5, the derivative $\tilde{\theta}'(\eta)$ is continuous, with n = 2 it undergoes a discontinuity of the first kind, and, with n = 3, reverts to infinity (see Fig. 3). However, for any given η, $0 \leq \eta < \infty$, there is a discontinuity of $\tilde{\theta}(\eta)$ and $[\tilde{\theta}^n(\eta)]'$, which corresponds to a discontinuity of the temperature and the heat flux. The expression for $\tilde{T}(x, t)$ with n > 1 is written in the form

$$\tilde{T}(x, y) = \begin{cases} T_w\left[1 - \frac{1}{4}(n-1)\sigma_w F_0\left(\frac{E}{\nu^3 x^3}\right)^{1/4} y\right]^{\frac{1}{n-1}}, & 0 \leqslant y \leqslant \tilde{y}_f, \\ 0, & \tilde{y}_f \leqslant y < \infty, \end{cases} \tag{11}$$

here the surface

$$y = \tilde{y}_f(x) = 4\nu^{3/4}x^{3/4}/(n-1)\sigma_w F_0 E^{1/4} \tag{12}$$

strictly bounds the regions in which $\tilde{T}(x, t) \neq 0$ and $\tilde{T}(x, y) = 0$, i.e., there is spatial localization of the thermal perturbations.

At the limit with $n \to 1 \pm 0$, from (11), (10), respectively, it follows that

$$\tilde{T}(x, y) = T_w \exp[-\sigma_w F_0 E^{1/4}/4(\nu x)^{3/4}], \; x, y \geqslant 0, \; n = 1. \tag{13}$$

In addition, as follows from (12), $\tilde{y}_f(x) \to \infty$ with $n \to 1 + 0$, i.e., with n = 1, as with n < 1, thermal perturbations exist everywhere in the region x, y ≥ 0.

The heat flux from the heated surface $\tilde{q}(x, 0) = -a(\partial\tilde{T}^n/dy)(x, 0)$ $(x \geq 0)$ is calculated using the expressions (10), (11), (13); here it is found that the value of $\tilde{q}(x, 0) = T_w(\nu E)^{1/4} \cdot F_0/4x^{3/4}$ does not depend on n.

It can be verified that, with n > 1, the heat flux is equal to zero everywhere at the surface $y = \tilde{y}_f(x)$; $(\tilde{\theta}^n)'(\tilde{\eta}_f) = 0$. Thus, the conditions $\tilde{\theta} = (\tilde{\theta}^n)' = 0$ (6) in the case n > 1, are found to be satisfied with a finite value of $\eta = \tilde{\eta}_f < \infty$, and not with $\eta = \infty$, as in the case n ≤ 1. Precisely this fact reflects the spatial localization of the thermal perturbations with n > 1. It must be noted that the above-mentioned localization is in no way due to the transition from problem (5) to the auxiliary problem (6). It can be shown that there is localization of the thermal perturbations if n > 1 in the equation of thermal conductivity (1).

We shall establish in this connection that, with $n > 1$, there exists a region $0 < x < \infty$, $0 < y_f(x) \le y < \infty$ ($y_f(x) = \eta_f (vx)^{3/4}/E^{1/4}$, $\eta_f = \text{const} > 0$), into which thermal perturbations from the heated surface do not penetrate. The latter assertion is equivalent to the fact that, with $n > 1$, there exists a set of values of $\eta: \eta_f \le \eta < \infty$, with which the solution of Eq. (5) reverts to zero.

In actuality, the solution $\theta(\eta)$ of the problem (5) with $0 < \eta_* < \infty$ ($\eta_* > 0$ is a fixed value of η) can be regarded as a solution of the problem

$$-F(\eta)\theta'(\eta) = a[\theta^n(\eta)]'', \quad \theta(\eta_*) = \theta_* \geqslant 0, \; \theta(\infty) = (\theta^n)'(\infty) = 0. \tag{14}$$

With $\eta > \eta_*$, the inequality holds

$$\theta(\eta) \leqslant \bar{\theta}(\eta), \tag{15}$$

where $\bar{\theta}(\eta)$ is a solution to the analogous (14) problem with a constant coefficient $F(\eta) \equiv F(\eta_*) = \text{const}$

$$-F(\eta_*)\bar{\theta}'(\eta) = a[\bar{\theta}^n(\eta)]'', \qquad \bar{\theta}(\eta_*) = \theta_*, \; \bar{\theta}(\infty) = (\bar{\theta}^n)'(\infty) = 0 \tag{16}$$

[$F(\eta_*) < F(\eta)$ with $\eta > \eta_*$ (see Fig. 2)]; i.e., the solution of problem (16) is a majorant of the solution of problem (14).

Actually, we denote $\psi_1(\eta) = \bar{\theta}^n(\eta)$, $\psi_2(\eta) = \theta^n(\eta)$, $f(\eta) = F(\eta)/a$, $\beta = 1/n$ and write Eqs. (14), (15) in the form

$$\psi_2''(\eta) = -f(\eta)\left[\psi_2^{\beta}(\eta)\right]', \quad \psi_1''(\eta) = -f(\eta_*)\left[\psi_1^{\beta}(\eta)\right]'. \tag{17}$$

We note first of all that it is sufficient to prove the inequality (15) only for values of η for which $\psi_2(\eta) > 0$ (since, while ψ_2 reverts to zero at some $\eta = \eta_f$, for $\eta > \eta_f$ it is continued by zero, i.e., by the singular solution of Eq. (14), and the inequality (15) is satisfied trivially). From the general theory of ordinary differential equations [6] it follows that, for all such values of η, the solution $\psi_2(\eta)$ of Eq. (17), satisfying the condition $\psi_2(\eta_*) = \theta_*^n$ exists.

Further, since $\psi_2(\eta) > 0$, the functions $\psi_2'(\eta)$ and $\psi_2''(\eta)$ have different signs. In addition, the function $\psi_2'(\eta)$ is continuous (this follows from the continuity of the heat flux). Therefore, from the condition $\psi_2(\infty) = 0$ it follows that the function $\psi_2(\eta)$ decreases monotonically with respect to η: i.e., $\psi_2'(\eta) \le 0$ with all $\eta > 0$; the sign of equality is possible only with $\psi_2(\eta) = 0$ (the function $\psi_1(\eta)$ of course has the same properties).

Setting $\alpha(\eta) = \psi_2(\eta) - \psi_1(\eta)$, we obtain

$$\begin{gathered}\alpha''(\eta) + f(\eta_*)\left[\psi_2^{\beta}(\eta) - \psi_1^{\beta}(\eta)\right]' = -[f(\eta) - f(\eta_*)]\left[\psi_2^{\beta}(\eta)\right]', \\ \alpha(\eta_*) = 0, \; \alpha(\infty) = \alpha'(\infty) = 0.\end{gathered} \tag{18}$$

Transforming $\psi_2^{\beta} - \psi_1^{\beta}$ in accordance with the Lagrange formula, and integrating (18) taking account of the conditions at infinity, we will have

$$\alpha'(\eta) + p(\eta)\alpha(\eta) = \varphi(\eta),$$

where $p(\eta) = \beta f(\eta_*)[\gamma\psi_1 + (1-\gamma)\psi_2]^{\beta-1}$, $0 < \gamma < 1$;

$$\varphi(\eta) = \int_{\eta}^{\infty} [f(\eta) - f(\eta_*)]\left[\psi_2^{\beta}(\eta)\right]' d\eta.$$

The solution of the equation obtained, satisfying the condition $\alpha(\eta_*) = 0$, can be written in the form

$$\alpha(\eta) = \exp\left[-\int_{\eta_*}^{\eta} p(\eta_1)\,d\eta_1\right]\left\{\int_{\eta_*}^{\eta} \varphi(\eta_2)\exp\left[\int_{\eta_*}^{\eta_2} p(\eta_3)\,d\eta_3\right]d\eta_2\right\}. \tag{19}$$

Under these circumstances, the function $\alpha(\eta)$ is bounded, since the functions $\psi_1(\eta)$ and $\psi_2(\eta)$ have this property. Thus, the solution of the problem (18), where it exists (i.e., where $\psi_1(\eta) > 0$ and $\psi_2(\eta) > 0$), can be represented in the form (19).

Taking into consideration that $\varphi(\eta) \le 0$ [since $\psi_2'(\eta) \le 0$[, from (19), we obtain $\alpha(\eta) \le 0$, which proves inequality (15).

If $\theta_* = 0$, then problem (16) has only the trivial solution $\bar{\theta}(\eta) \equiv 0$, $\eta_* \leq \eta < \infty$; therefore, $\eta_f = \eta_* \leq \infty$.

If $\theta_* > 0$, then the solution of problem (16) is constructed analogously to solution (11) of the auxiliary problem (6); it is written in the form

$$\bar{\theta}(\eta) = \begin{cases} \theta_* \left[1 - \frac{(n-1)F(\eta_*)}{an\theta_*^{n-1}}(\eta - \eta_*)\right]^{\frac{1}{n-1}}, & \eta_* \leqslant \eta \leqslant \bar{\eta}_f, \\ 0, & \bar{\eta}_f \leqslant \eta < \infty, \end{cases}$$

$$\bar{\eta}_f = \eta_* + an\theta_*^{n-1}/((n-1)F(\eta_*)),$$

from which, by virtue of (15), it follows that $\eta_f \leq \bar{\eta}_f < \infty$.

Thus, with $n > 1$, in a pre-wall jet there always exists a region bounded by the surface $y = y_f(x)$, into which thermal perturbations from the heated wall do not penetrate. In the general case, it is possible to construct a qualitative description of the behavior of the function $T(x, y)$ near this surface. For such a description, it is sufficient to limit the discussion of the thermal-conductivity equation (1) to the small neighborhood of some fixed point x_0, y_0 on the curve $y = y_f(x)$. Introducing the local coordinates at the point x_0, y_0

$$\xi(x, y) = [k(x - x_0) - (y - y_0)]/(1 + k^2)^{1/2},$$
$$\zeta(x, y) = [(x - x_0) + k(y - y_0)]/(1 + k^2)^{1/2},$$
$$k = (dy_f/dx)(x_0)$$

[normal and tangential to the surface $y = y_f(x)$, respectively], we write in them the thermal-conductivity equation (1), taking into consideration that, in a small neighborhood of the point x_0, y_0, the functions $u(x, y)$ and $v(x, y)$ can be replaced by the constants $u_0 = u(x_0, y_0)$ and, $v(x_0, y_0)$, and that the temperature depends essentially only on ξ. As a result, we have

$$-(1 + k^2)^{1/2}(ku_0 - v_0)\frac{dT}{d\xi} = a\frac{d^2T^n}{d\xi^2}. \tag{20}$$

Taking account of expression (2), by an indirect verification it can be demonstrated that $udy_f/dx - v > 0$ and, consequently $ku_0 - v_0 > 0$. As a result of a double integration of Eq. (20), taking account of the conditions at the surface $y = y_f(x)$, we obtain the expression

$$T(\xi) = [(n-1)(ku_0 - v_0)(1 + k^2)^{1/2}(-\xi)/na]^{1/(n-1)},$$

determining the distribution of the temperature near the surface $\xi = 0$ with $\xi < 0$. With $\xi > 0$, the distribution of the temperature must be determined by the singular solution of Eq. (20) ($T(\xi) \equiv 0$), which exists only with $n > 1$ [6].

In [1] it was shown that, in media with thermal-conductivity coefficient anT^{n-1} ($n > 1$), the thermal perturbations are propagated with a finite velocity. The finite nature of the velocity of the propagation of the thermal perturbations and the deflection of the liquid in the pre-wall jet are the main reasons for the existence of a stationary surface in the case in question, i.e., the front of a "thermal wave," separating regions with $T \neq 0$ and $T = 0$. As has been noted in [7], the existence of a "thermal wave" front is connected with the singular solution of an ordinary differential describing the self-similar distribution of the temperature.

In conclusion we note that all the results established above automatically carry over to the corresponding problem of nonlinear diffusion.

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ASYMPTOTIC OF SOLUTION OF PROBLEM OF CONVECTIVE DIFFUSION TO A DROP WITH LARGE PÉCLET NUMBERS AND FINITE REYNOLDS NUMBERS

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A first approximation in the problem of steady-state convective diffusion to a spherical particle in a homogeneous translational flow has been obtained for zero [1] and finite Reynolds numbers [2, 3]. A two-term expansion in the case of Stokes flow around a solid particle is given in [4].

We postulate that the concentration of the substance dissolved in the flow is constant far from the drop, and that it is completely absorbed at the surface. In a spherical system of coordinates connected with the drop, the dimensionless equation of convective diffusion and the boundary conditions have the form (Pe is the Péclet number)

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r}\frac{\partial c}{\partial \theta} = e^2\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right)\right\},$$
$$r=1,\quad c=0;\quad r=\infty,\quad c=1, \tag{1}$$
$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\quad e^2 = \mathrm{Pe}^{-1} = \frac{D}{aU}.$$

Here the concentration at infinity, the velocity of the oncoming flow U, and the radius of the drop a are taken as the scales of the concentration, the velocity, and the length; the angle θ is reckoned from the direction of the flow at infinity.

For the field of the velocities we use expressions obtained for a drop by the method of joined asymptotic expansions [5]:

$$\psi = \psi_0 + \mathrm{Re}\psi_1 \quad (\mathrm{Re} = aU/\nu),$$
$$\psi_0 = \frac{1}{2}(r-1)\left[r - \frac{1}{2}\frac{\beta}{\beta+1}\left(1+\frac{1}{r}\right)\right]\sin^2\theta, \tag{2}$$
$$\psi_1 = \frac{1}{8}\frac{3\beta+2}{\beta+1}\psi_0 - \frac{1}{16}\frac{3\beta+2}{\beta+1}(r-1)\left[r - \frac{1}{2}\frac{\beta}{\beta+1} - \frac{1}{10}\frac{\beta}{(\beta+1)^2}\left(\frac{1}{r} + \frac{5\beta+6}{r^2}\right)\right]\sin^2\theta\cos\theta,$$

where β is the ratio of the viscosities of the drop and the liquid surrounding it; Re is the Reynolds number.

We shall assume that the Péclet number is large (the parameter e is small); we introduce the extended coordinate Y in the diffusional boundary layer and represent the flow function (2) in the form of a series

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